CWI Quarterly Volume 9 (1 & 2) 1996, pp. 37 – 40

Spectrum and Connectivity of Graphs

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PROBLEM. Let Γ be a nice graph. Show that Γ is very connected.

In this talk I would like to give three examples of results about the connectivity of a graph that follow by considering its spectrum. There are lots of open problems.

Three measures of connectivity play a rôle here:

(i) is Γ connected or not?

(ii) $\kappa(\Gamma)$, the vertex connectivity of Γ, that is, the minimum number of vertices that one has to remove in order to disconnect Γ.

(iii) $t(\Gamma)$, the toughness of Γ, is defined as

$$\min_S \frac{|S|}{c(\Gamma \setminus S)}$$

where S runs over all sets of vertices such that $\Gamma \setminus S$ is disconnected, and $c(\Gamma \setminus S)$ is its number of connected components.

The graph K_0 without vertices is not connected (we have $c(K_0) = 0$, while $c(\Gamma) = 1$ for connected graphs Γ) but I shall leave undefined whether it is disconnected, and hence do not define $\kappa(\Gamma)$ and $t(\Gamma)$ when Γ is complete.

For example, for the Petersen graph we find $\kappa(\Gamma) = 3$ and $t(\Gamma) = \frac{4}{3}$. More generally, we clearly have $\kappa(\Gamma) \le k(\Gamma)$ if $k(\Gamma)$ is the (minimal) valency of Γ.

One may also ask about the size of 'nonlocal' cut sets. For example,

(1) ('unimodality') Is it true that if S is a cut set of Γ, with separation $\Gamma \setminus S = A + B$, then $\min(|\Gamma(S) \cap A|,\ |\Gamma(S) \cap B|) \le |S|$? (Here $\Gamma(S)$ denotes the set of all vertices adjacent to some vertex of S.) (Jack Koolen remarks that some condition is necessary: for each i, $0 \le i \le 17$, the Biggs-Smith graph has a cut set S of size 17 such that $|\Gamma(S) \cap A| = 17+i$, $|\Gamma(S) \cap B| = 34-i$.)

(2) Show that $|S|$ is substantially larger than k when S is nonlocal (say, given a lower bound on the size or the minimum valency of each component of $\Gamma \setminus S$).

1. The connectivity of strongly regular graphs

Theorem 1.1. (Brouwer & Mesner [4]) *Let Γ be strongly regular of valency k. Then $\kappa(\Gamma) = k$, and the only cut sets of size k are the point neighbourhoods.*

The weaker result that for strongly regular graphs the edge-connectivity is at least the valency, was shown ten years earlier by Plesník[10].

Open problems are for example:

(3) Prove the above result for distance-regular graphs.
(4) Let Γ be strongly regular with parameters (v, k, λ, μ), and let S be a disconnecting set not containing any point neighbourhood $\Gamma(x)$. Show that $|S| \geq 2k - 2 - \lambda$.
(5) Let S be a disconnecting set such that $|S \cap \Gamma(x)| \leq \alpha k$ for some fixed α, $0 < \alpha < 1$, and all vertices x of Γ. Prove a superlinear (in k) lower bound for $|S|$.

Godsil [7] conjectures that for any graph that is a colour class in an association scheme, the edge-connectivity equals the degree, and I conjecture the same for the vertex-connectivity.

Partial results: Fiedler [6] shows that the vertex-connectivity of a graph is at least $k - \theta_2$, where k is the degree and θ_2 the second largest eigenvalue. Lovász [9], Problem 12.14, proves that the edge-connectivity of a vertex-transitive graph is at least its degree. An old result says that the vertex-connectivity of a vertex-transitive graph is at least $2(k+1)/3$, where the extreme case is the clique extension of a circuit.

Brouwer & Mulder [5] showed $\kappa(\Gamma) = k$ for graphs with the property that any two distinct vertices have either 0 or 2 common neighbours. This settles (3) in the case $\lambda \in \{0, 2\}$, $\mu = 2$.

2. The connectedness of generic pieces of generalized polygons

Theorem 2.1. (Brouwer [2]) *Let Γ be the point graph or the flag graph of a finite generalized polygon. Then the subgraph Δ of Γ induced on the set of all vertices far away from ('in general position w.r.t.') a point or flag is connected, except in the cases $G_2(2)$, ${}^2F_4(2)$ and (for the flag graph) $B_2(2)$, $G_2(3)$. A similar result holds more generally for the complement of a geometric hyperplane.*

Open problems:

(6) Generalize this to near polygons.
(7) Generalize this to distance-regular graphs.

It is very easy to see that in a strongly regular graph the subgraph on the vertices far away from a point is connected (except when the graph is complete multipartite).

3. The toughness of a regular graph

Theorem 3.1. (Alon-Brouwer, *cf. [1, 3]) Let Γ be a graph on v vertices, regular of valency k, and with eigenvalues $k = \theta_1 \geq \theta_2 \geq \ldots \geq \theta_v$. Put*

$$\lambda := \max_{2 \leq j \leq v} |\theta_j|.$$

Then

$$t(\Gamma) > \frac{k}{\lambda} - 2.$$

Open problems:

(8) Prove $t(\Gamma) \geq \frac{k}{\lambda} - 1$. (I conjecture that this is the right bound.)
(9) Prove $t(\Gamma) = \frac{k}{\lambda}$ in many cases.

Examples We have bipartite graphs of small toughness, so the '-1' would be best possible. The Delsarte-Hoffman bound for cocliques C in strongly regular graphs states

$$|C| \leq \frac{v}{1 + k/(-\theta_v)}.$$

If equality holds, and $\lambda = -\theta_v$ (as is often the case), then we find with $S = \Gamma \backslash C$: $t(\Gamma) \leq (v - |C|)/|C| = \frac{k}{\lambda}$.

4. Tools

How are these results proved? Essentially, only interlacing (cf. Haemers [8]) is used. Interlacing comes in two main forms:

(i) If Δ is an induced subgraph of a graph Γ, then the eigenvalues η_j ($1 \leq j \leq u$) of Δ interlace the eigenvalues θ_i ($1 \leq i \leq v$) of Γ: we have $\theta_i \geq \eta_i$ ($1 \leq i \leq u$) and $\eta_{u-j} \geq \theta_{v-j}$ ($0 \leq j \leq u-1$).

(ii) Given a partition Π of the index set of a symmetric matrix A, let $B = (B_{R,S})_{R,S \in \Pi}$ be the matrix of average row sums of the corresponding submatrices of A. Then the eigenvalues of B interlace those of A.

Examples

Lemma 4.1. *The average valency of a graph is not more than its largest eigenvalue.*

Proof. Use a partition with 1 part. □

Lemma 4.2. *Let Γ be regular of valency k on v vertices, and let the graph induced on the r-set R have average valency k_R. Then*

$$\theta_2 \geq (vk_R - rk)/(v - r) \geq \theta_v,$$

(and hence

$$r \leq v(k_R - \theta_v)/(k - \theta_v).$$

For $k_R = 0$ we find the Delsarte-Hoffman bound).

PROOF. Use a partition with 2 parts. □

LEMMA 4.3. *Let* Γ *and* R *be as before. Put* $\lambda = \max(|\theta_2|, |\theta_v|)$. *Then*

$$\sum_x (|\Gamma(x) \cap R| - \frac{rk}{v})^2 \leq \lambda^2 r(v-r)/v.$$

PROOF. Use a partition with 2 parts, and apply to A^2, the square of the adjacency matrix of Γ. □

5. Proofs of the results in sections 1,2,3

Let Γ have eigenvalues $k = \theta_1 \gg \theta_2 \geq \ldots$. If Δ is a disconnected subgraph, then its spectrum is the union of the spectra of its components. Each component has a largest eigenvalue at least as large as its average degree, and by interlacing it follows that all components except perhaps one have average degree at most θ_2, but this is much too small (except when Γ is very small).

This proves the results of Sections 1 and 2. For those of Section 3, use the above three Lemmata and compute.

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